A W*-correspondence approach to multidimensional linear dissipative systems

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Joint work with J.A. Ball

Noncommutative multidimensional linear dissipative systems

Consider a linear system evolving over $\mathcal{F}_d$; the free semigroup of words $\alpha = i_{k_n} \cdots i_{k_1}$ generated by d letters $\{i_1, \ldots, i_d\}$ with neutral element $\emptyset$:

$$\left\{ \begin{array}{rcl} x(i_k \cdot \alpha) & = & A_k x(\alpha) + B_k u(\alpha), \;\; k = 1, \ldots, d, \\ y(\alpha) & = & Cx(\alpha) + Du(\alpha), \end{array} \right. \qquad (\alpha \in \mathcal{F}_d)$$

with contractive system matrix

$$\begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix}.$$

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When $x(\emptyset) = 0$ and $u = (u(\alpha))_{\alpha \in \mathcal{F}_d}$ is in $\ell^2_{\mathcal{U}}(\mathcal{F}_d)$, then $y = (y(\alpha))_{\alpha \in \mathcal{F}_d}$ is in $\ell^2_{\mathcal{Y}}(\mathcal{F}_d)$ and the input-output map in the frequency domain is given by

$$\hat{y}(z) = T_\Sigma(z)\hat{u}(z)$$

where the *transfer function* $T_\Sigma(z)$ and the *Z-transforms* $\hat{u}(z)$ and $\hat{y}(z)$ are given by formal power series in noncommuting indeterminates $z = (z_1, \ldots, z_d)$:

$$T_\Sigma(z) = D + C(I - \textstyle\sum_{k=1}^d z_k A_k)^{-1}(\sum_{i=1}^d z_i B_i),$$

$$\hat{u}(z) = \textstyle\sum_{\alpha \in \mathcal{F}_d} z^\alpha u(\alpha), \quad \hat{y}(z) = \sum_{\alpha \in \mathcal{F}_d} z^\alpha y(\alpha),$$

where $z^\alpha = z_{k_n} \cdots z_{k_1}$ in case $\alpha = i_{k_n} \cdots i_{k_1}$.

Commutative multidimensional linear dissipative systems

Consider a linear system evolving over $\mathbb{Z}_+^d$:

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where $e_1 = (1, 0, \dots, 0), \dots, e_d = (0, \dots, 0, 1)$ and we set $x(\mathbf{n}) = 0$ and $u(\mathbf{n}) = 0$ for $\mathbf{n} \in \mathbb{Z}^d - \mathbb{Z}_+^d$, with contractive system matrix

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When $x(\mathbf{n}) = 0$ for $\mathbf{n} = (0, \dots, 0)$ and $u = (w(\mathbf{n})u(\mathbf{n}))_{\mathbf{n} \in \mathbb{Z}_+^d}$ is in $\ell^2_{\mathcal{U}}(\mathbb{Z}_+^d)$, then $y = (w(\mathbf{n})y(\mathbf{n}))_{\mathbf{n} \in \mathbb{Z}_+^d}$ is in $\ell^2_{\mathcal{Y}}(\mathbb{Z}_+^d)$, where $(w(\mathbf{n}))_{\mathbf{n} \in \mathbb{Z}_+^d}$ is some weighting sequence, and:

$$\hat{y}(\lambda_1, \dots, \lambda_d) = T_\Sigma(\lambda_1, \dots, \lambda_d)\hat{u}(\lambda_1, \dots, \lambda_d) \quad (|\lambda_1|^2 + \cdots + |\lambda_d|^2 < 1)$$

where the *transfer function* T_Σ and the *Z-transforms* $\hat{u}$ and $\hat{y}$ are given by

$$T_\Sigma(\lambda_1, \dots, \lambda_d) = D + C(I - \textstyle\sum_{k=1}^d \lambda_k A_k)^{-1}(\sum_{i=1}^d \lambda_i B_i)$$

$$\hat{u}(\lambda_1, \dots, \lambda_d) = \textstyle\sum_{(n_1, \dots, n_d) \in \mathbb{Z}_+^d} \lambda_1^{n_1} \cdots \lambda_d^{n_d} u(n_1, \dots, n_d) \qquad (|\lambda_1|^2 + \cdots + |\lambda_d|^2 < 1).$$

$$\hat{y}(\lambda_1, \dots, \lambda_d) = \textstyle\sum_{(n_1, \dots, n_d) \in \mathbb{Z}_+^d} \lambda_1^{n_1} \cdots \lambda_d^{n_d} y(n_1, \dots, n_d)$$

W^*-correspondences

C^*-correspondences

Given C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, a linear space E is an $(\mathcal{A}, \mathcal{B})$-correspondence when E is a bi-module with a left $\mathcal{A}$-action and right $\mathcal{B}$-action, with a $\mathcal{B}$-valued inner product satisfying the following axioms: For $\xi, \zeta, \eta \in E$, $\lambda, \mu \in \mathbb{C}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$

- $\langle \lambda\xi + \mu\zeta, \eta\rangle_E = \lambda \langle \xi, \eta\rangle_E + \mu \langle \zeta, \eta\rangle_E$;
- $\langle \xi \cdot b, \eta\rangle_E = \langle \xi, \eta\rangle_E\, b$; $\qquad \langle a \cdot \xi, \eta\rangle_E = \langle \xi, a^* \cdot \eta\rangle_E$;
- $\langle \xi, \eta\rangle_E^* = \langle \eta, \xi\rangle_E$;
- $\langle \xi, \xi\rangle_E \geq 0$; $\quad \langle \xi, \xi\rangle_E = 0$ implies that $\xi = 0$;

and such that E is a Banach space with respect to the norm $\|\xi\|_E := \| \langle \xi, \xi\rangle_E \|_{\mathcal{B}}^{\frac{1}{2}}$ for $\xi \in E$, where $\| \ \|_{\mathcal{B}}$ denotes the norm of $\mathcal{B}$.

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Notation and terminology

Given two $(\mathcal{A},\mathcal{B})$-correspondences E_1 and E_2:

- $\mathcal{L}^a(E_1,E_2)$ denotes the set of bounded linear *adjointable* operators $T : E_1 \to E_2$ (i.e. $\langle T\xi_1,\xi_2\rangle_{E_2} = \langle \xi_1, T^*\xi_2\rangle_{E_1}$ for some $T \in \mathcal{L}(E_2,E_1)$).
- An operator $T \in \mathcal{L}^a(E_1,E_2)$ is called an $\mathcal{A}$-*module map* if

$$T(a\xi) = a(T\xi) \quad (\xi \in E_1, a \in \mathcal{A}).$$

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- $\langle \lambda\xi + \mu\zeta, \eta \rangle_E = \lambda \langle \xi, \eta \rangle_E + \mu \langle \zeta, \eta \rangle_E$;
- $\langle \xi \cdot b, \eta \rangle_E = \langle \xi, \eta \rangle_E \, b$; $\qquad \langle a \cdot \xi, \eta \rangle_E = \langle \xi, a^* \cdot \eta \rangle_E$;
- $\langle \xi, \eta \rangle_E^* = \langle \eta, \xi \rangle_E$;
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W^*-correspondences

An $(\mathcal{A},\mathcal{B})$-correspondence E is a W^*-$(\mathcal{A},\mathcal{B})$-correspondence if in addition

- $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras;
- $T \in \mathcal{L}^a(E,\mathcal{B}) \implies$ there exists a $\xi_T \in E$ so that $T\xi = \langle \xi, \xi_T \rangle$ for all $\xi \in E$.

Examples of W^*-correspondences

1. Any Hilbert space is a W^*-$(\mathbb{C}, \mathbb{C})$-correspondence.
2. $\mathcal{A}\mathcal{B} = E = \mathcal{L}(\mathcal{U})$, for some Hilbert space $\mathcal{U}$, is a $(\mathcal{L}(\mathcal{U}), \mathcal{L}(\mathcal{U}))$-correspondence with inner product $\langle a, b\rangle = b^* a$. The generating bounded linear operator on $\mathcal{L}(\mathcal{U})$:
$$K \mapsto T_1 K T_2 \quad \text{for given } T_1, T_2 \in \mathcal{L}(\mathcal{U})$$
   - is adjointable $\iff$ $T_2 = \lambda I_{\mathcal{U}}$ for some $\lambda \in \mathbb{D}$;
   - is a $\mathcal{L}(\mathcal{U})$-module map $\iff$ $T_1 = \lambda I_{\mathcal{U}}$ for some $\lambda \in \mathbb{D}$.
3. Main example: $\mathcal{A} = \mathcal{B} = \mathcal{L}(\mathcal{U})$ and $E = \mathcal{L}(\mathcal{U}, \mathcal{U}^d)$, operator columns of length d, subject to
$$A_1 \cdot \begin{bmatrix} T_1 \\ \vdots \\ T_d \end{bmatrix} \cdot A_2 = \begin{bmatrix} A_1 T_1 A_2 \\ \vdots \\ A_1 T_d A_2 \end{bmatrix}, \quad \left\langle \begin{bmatrix} T_1 \\ \vdots \\ T_d \end{bmatrix}, \begin{bmatrix} S_1 \\ \vdots \\ S_d \end{bmatrix} \right\rangle = \sum_{k=1}^{d} S_k^* T_k.$$
4. Other examples:
   - systems evolving over quiver algebras;
   - timevarying systems (Alpay-Ball-Peretz '02);
   - analytic crossed-product algebras.

Constructing new correspondences

Direct sum correspondences

Similar to the Hilbert space case: Given two $(\mathcal{A}, \mathcal{B})$-correspondences E and F, we can form a *direct-sum* $(\mathcal{A}, \mathcal{B})$*-correspondence* $E \oplus F$.

Tensor product correspondences

Given an $(\mathcal{A}, \mathcal{B})$-correspondence E and a $(\mathcal{B}, \mathcal{C})$-correspondence F, we form

$$E \otimes F = \overline{\operatorname{span}}\{\xi \otimes \gamma \colon \xi \in E, \gamma \in F\}$$

where we identify

$$(\xi \cdot b) \otimes \gamma = \xi \otimes (b \cdot \gamma).$$

Then $E \otimes F$ is a *(tensor product)* $(\mathcal{A}, \mathcal{C})$*-correspondence* subject to

$$a \cdot (\xi \otimes \gamma) = (a \cdot \xi) \otimes \gamma, \qquad (\xi \otimes \gamma) \cdot c = \xi \otimes (\gamma \cdot c),$$
$$\langle \xi \otimes \gamma, \xi' \otimes \gamma' \rangle_{E \otimes F} = \left\langle \langle \xi, \xi' \rangle_E \cdot \gamma, \gamma' \right\rangle_F .$$

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Operators between tensor product correspondences

For $T \in \mathcal{L}^a(E_1, E_2)$ and $S \in \mathcal{L}^a(F_1, F_2)$ with S a $\mathcal{B}$-module map we define $T \otimes S \in \mathcal{L}^a(E_1 \otimes F_1, E_2 \otimes F_2)$ by

$$(T \otimes S)(\xi \otimes \eta) = (T\xi) \otimes (S\eta) \quad (\xi \in E_1, \eta \in F_1).$$

Correspondence-representation pairs and their duals

CR-pairs

A *correspondence-representation pair* (CR-pair) is a pair (E, σ) consisting of:

- a W^*-$(\mathcal{A}, \mathcal{A})$-correspondence E;
- a non-degenerate $*$-homomorphism $\sigma : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, $\mathcal{H}$ some Hilbert space.

Then $\mathcal{H}$ is an $(\mathcal{A}, \mathbb{C})$-correspondence with left $\mathcal{A}$-action given by σ.

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Dual CR-pairs

Given a CR-pair (E, σ) define

$$E^\sigma := \{\eta : \mathcal{H} \to E \otimes \mathcal{H} \colon \eta \text{ is an } \mathcal{A}\text{-module map}\}$$
$$\sigma(\mathcal{A})' := \{b \in \mathcal{L}(\mathcal{H}) \colon b\sigma(a) = \sigma(a)b \quad (a \in \mathcal{A})\}.$$

THM.(Muhly-Solel, 2004) *E^σ is a W^*-$(\sigma(\mathcal{A})', \sigma(\mathcal{A})')$-correspondence with:*

$$b_1 \cdot \eta \cdot b_2 = (I_E \otimes b_1)\eta b_2 \quad \langle \eta', \eta \rangle = \eta^* \eta' \quad (\eta, \eta' \in E^\sigma, b_1, b_2 \in \sigma(\mathcal{A})').$$

Together with identity representation $\iota : \sigma(\mathcal{A})' \to \mathcal{L}(\mathcal{H})$, $\iota(b) = b$, the pair (E^σ, ι) forms a CR-pair. Finally, for $n = 0, 1, 2, \ldots$ there exists a unitary map

$$\Phi_n : (E^\sigma)^{\otimes n} \otimes \mathcal{H} \to E^{\otimes n} \otimes \mathcal{H},$$
$$\Phi_n(\eta_n \otimes \cdots \otimes \eta_1 \otimes u) = (I_{E^{\otimes n-1}} \otimes \eta_n) \cdots (I_E \otimes \eta_2)\eta_1 u,$$

where $E^{\otimes 0} = \mathcal{A}$, $E^{\otimes 1} = E$, and $E^{\otimes n+1} = E \otimes E^{\otimes n}$, and similarly for E^σ.

Main example revisited (1)

Take $\mathcal{A} = \mathcal{L}(\mathcal{U})$, $E = \mathcal{L}(\mathcal{U}, \mathcal{U}^d)$.

- CR-pair (E, σ): $\sigma : \mathcal{A} \to \mathcal{L}(\mathcal{U} \otimes \mathcal{K})$, $\sigma(A) = A \otimes I_{\mathcal{K}}$, for some fixed Hilbert space $\mathcal{K}$.
- Special case: $\mathcal{K} = \mathbb{C}$ (commutative) and $\mathcal{K} = \ell^2(\mathbb{Z})$ (noncommutative).

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- Dual of (E, σ):

$$\sigma(\mathcal{A})' = \{I_{\mathcal{U}} \otimes M \colon M \in \mathcal{L}(\mathcal{K})\} \cong \mathcal{L}(\mathcal{K});$$

$$E^{\sigma} = \left\{ \begin{bmatrix} I_{\mathcal{U}} \otimes T_1 \\ \vdots \\ I_{\mathcal{U}} \otimes T_d \end{bmatrix} : T_1, \dots, T_d \in \mathcal{L}(\mathcal{K}) \right\} \cong \mathcal{L}(\mathcal{K}, \mathcal{K}^d);$$

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- Then

$$E^{\otimes n} = \mathcal{L}(\mathcal{U}, \mathcal{U}^{(d^n)}), \qquad E^{\sigma \otimes n} \cong \mathcal{L}(\mathcal{K}, \mathcal{K}^{(d^n)}),$$

$$E^{\otimes n} \otimes (\mathcal{U} \otimes \mathcal{K}) \cong (\mathcal{U} \otimes \mathcal{K})^{(d^n)}, \quad E^{\sigma \otimes n} \otimes (\mathcal{U} \otimes \mathcal{K}) \cong (\mathcal{U} \otimes \mathcal{K})^{(d^n)}.$$

Note: $d^n = \#$words in $\{1, \dots, d\}$ of length n.

Dissipative systems

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$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{H} \end{bmatrix} \to \begin{bmatrix} E^\sigma \otimes \mathcal{X} \\ \mathcal{H} \end{bmatrix}$$

where $\mathcal{X}$ is some $(\sigma(\mathcal{A})', \mathbb{C})$-correspondence (i.e., a Hilbert space with a left $\sigma(\mathcal{A})'$-action) and A, B, C and D are $\sigma(\mathcal{A})'$-module maps.

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The system equations corresponding to this system matrix are given by

$$\left\{ \begin{array}{rcl} x(n+1) & = & A_n x(n) + B_n \Phi_n^* u(n), \\ y(n) & = & \Phi_n C_n x(n) + \Phi_n D_n \Phi_n^* u(n), \end{array} \right. \qquad n = 0, 1, 2, \ldots$$

where

$$y(n), u(n) \in E^{\otimes n} \otimes \mathcal{H}, \quad x(n) \in (E^\sigma)^{\otimes n} \otimes \mathcal{X}$$

and

$$A_n := I_{(E^\sigma)^{\otimes n}} \otimes A, \quad B_n := I_{(E^\sigma)^{\otimes n}} \otimes B, \quad C_n := I_{(E^\sigma)^{\otimes n}} \otimes C, \quad D_n := I_{(E^\sigma)^{\otimes n}} \otimes D.$$

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When $x(0) = x_0 \in \mathcal{X}$ is fixed, then $x = (x(n))_{n \in \mathbb{Z}_+}$ and $y = (y(n))_{n \in \mathbb{Z}_+}$ are completely determined by the input sequence $u = (u(n))_{n \in \mathbb{Z}_+}$.

The Hilbert spaces $\mathcal{F}^2(E,\sigma)$ and $H^2(E,\sigma)$

Given a CR-pair (E,σ), we define the Fock space $(\mathcal{A},\mathbb{C})$-correspondence

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To make the transfer from the "time domain" to the "frequency domain" we introduce the *generalizes disk*:

$$\mathbb{D}((E^\sigma)^*) = \{\eta : E\otimes\mathcal{H}\to\mathcal{H} \colon \eta^* \in E^\sigma,\ \|\eta\|<1\}.$$

For $\eta\in\mathbb{D}((E^\sigma)^*)$ we define the *generalized powers*:

$$\eta^k := \eta(I_E\otimes\eta)\cdots(I_{E^{\otimes n-1}}\otimes\eta) : E^{\otimes k}\otimes\mathcal{H}\to\mathcal{H}, \text{ and set } \eta^0 = I_{\mathcal{H}}.$$

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$$\mathbb{D}((E^\sigma)^*) = \{\eta : E\otimes\mathcal{H}\to\mathcal{H} \colon \eta^*\in E^\sigma,\ \|\eta\|<1\}.$$

For $\eta\in\mathbb{D}((E^\sigma)^*)$ we define the *generalized powers*:

$$\eta^k := \eta(I_E\otimes\eta)\cdots(I_{E^{\otimes n-1}}\otimes\eta) : E^{\otimes k}\otimes\mathcal{H}\to\mathcal{H},\ \text{and set } \eta^0 = I_{\mathcal{H}}.$$

The *Z-transform* $f\mapsto\hat{f}$ in this setting sends an $f=(f_n)_{n\in\mathbb{Z}_+}\in\mathcal{F}^2(E,\sigma)$ to the function $\hat{f}:\mathbb{D}((E^\sigma)^*)\times\sigma(\mathcal{A})'\to\mathcal{H}$ given by

$$\hat{f}(\eta,b) = \sum_{k=0}^{\infty}\eta^k(I_{E^{\otimes k}}\otimes b)f_k.$$

The space $H^2(E,\sigma)$ consisting of such functions $\hat{f}$ is a Hilbert space (with $\|\hat{f}\| = \|f\|_{\mathcal{F}^2(E,\sigma)}$) with a left $\sigma(\mathcal{A})'$-action given by $(b'\hat{f})(\eta,b) = \hat{f}(\eta,bb')$.

Main result

THM.(Variation on Muhly-Solel '08; see also Ball-Biswas-Fang-tH '08)
Given a CR-pair (E,σ) *and a dissipative system*

$$\Sigma := \begin{cases} x(n+1) = A_n x(n) + B_n \Phi_n^* u(n), \\ y(n) = \Phi_n C_n x(n) + \Phi_n D_n \Phi_n^* u(n), \end{cases} \qquad n = 0, 1, 2, \ldots$$

with $x(0) = 0$, *and* $u = (u(n))_{n\in\mathbb{Z}_+} \in \mathcal{F}^2(E,\sigma)$, *then the output sequence* $y = (y(n))_{n\in\mathbb{Z}_+}$ *is in* $\mathcal{F}^2(E,\sigma)$, *and the* Z*-transforms* $\hat{u}$ *and* $\hat{y}$ *of* u *and* y *are related through the input-output map*

$$\hat{y}(\eta, b) = T_\Sigma(\eta)\hat{u}(\eta, b) \quad (\eta \in \mathbb{D}((E^\sigma)^*), b \in \sigma(\mathcal{A})'),$$

where the transfer function $T_\Sigma : \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(\mathcal{H})$ *is given by*

$$T_\Sigma(\eta) = D + C(I - L_{\eta^*}^* A)^{-1} L_{\eta^*}^* B$$

$$\text{with } L_{\eta^*} : \mathcal{X} \to E^\sigma \otimes \mathcal{X}, \quad L_{\eta^*} x = \eta^* \otimes x.$$

Moreover, T_Σ *defines a contractive multiplier on* $H^2(E,\sigma)$, *and all contractive multiplier on* $H^2(E,\sigma)$ *are obtained in this way.*

Main example revisited (2)

Take $\mathcal{A} = \mathcal{L}(\mathcal{U})$, $E = \mathcal{L}(\mathcal{U}, \mathcal{U}^d)$, $\sigma(A) = A \otimes I_{\mathcal{K}} \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K})$. Then a dissipative system matrix has the form

$$\begin{bmatrix} A_1 \otimes I_{\mathcal{K}} & B_1 \otimes I_{\mathcal{K}} \\ \vdots & \vdots \\ A_d \otimes I_{\mathcal{K}} & B_d \otimes I_{\mathcal{K}} \\ C \otimes I_{\mathcal{K}} & D \otimes I_{\mathcal{K}} \end{bmatrix} \cong \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{U} \end{bmatrix},$$

corresponding to a system

$$\Sigma := \begin{cases} x(n+1) = \tilde{A}_n x(n) + \tilde{B}_n u(n), \\ y(n) = \tilde{C}_n x(n) + \tilde{D}_n u(n), \end{cases} \qquad (n \in \mathbb{Z}_+)$$

with inputs, outputs and states $u(n), y(n) \in (\mathcal{U} \otimes \mathcal{K})^{(d^n)}$, $x(n) \in (\mathcal{X} \otimes \mathcal{K})^{(d^n)}$ and where

$$\tilde{A}_n = \mathrm{blockdiag}_{i=1,\dots,d^n}\left(\begin{bmatrix} A_1 \otimes I_{\mathcal{K}} \\ \vdots \\ A_d \otimes I_{\mathcal{K}} \end{bmatrix}\right), \quad \tilde{B}_n = \mathrm{blockdiag}_{i=1,\dots,d^n}\left(\begin{bmatrix} B_1 \otimes I_{\mathcal{K}} \\ \vdots \\ B_d \otimes I_{\mathcal{K}} \end{bmatrix}\right),$$

$$\tilde{C}_n = \mathrm{blockdiag}_{i=1,\dots,d^n}\left(\begin{bmatrix} C \otimes I_{\mathcal{K}} \end{bmatrix}\right), \quad \tilde{D}_n = \mathrm{blockdiag}_{i=1,\dots,d^n}\left(\begin{bmatrix} D \otimes I_{\mathcal{K}} \end{bmatrix}\right).$$

Main example revisited (2)

Take $\mathcal{A} = \mathcal{L}(\mathcal{U})$, $E = \mathcal{L}(\mathcal{U}, \mathcal{U}^d)$, $\sigma(A) = A \otimes I_{\mathcal{K}} \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K})$. Then a dissipative system matrix has the form

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corresponding to a system

$$\Sigma := \begin{cases} x(n+1) &= \tilde{A}_n x(n) + \tilde{B}_n u(n), \\ y(n) &= \tilde{C}_n x(n) + \tilde{D}_n u(n), \end{cases} \qquad (n \in \mathbb{Z}_+)$$

with inputs, outputs and states $u(n), y(n) \in (\mathcal{U} \otimes \mathcal{K})^{(d^n)}$, $x(n) \in (\mathcal{X} \otimes \mathcal{K})^{(d^n)}$ and where

$$\tilde{A}_n = \mathrm{blockdiag}_{i=1,\dots,d^n}\left(\begin{bmatrix} A_1 \otimes I_{\mathcal{K}} \\ \vdots \\ A_d \otimes I_{\mathcal{K}} \end{bmatrix} \right), \quad \tilde{B}_n = \mathrm{blockdiag}_{i=1,\dots,d^n}\left(\begin{bmatrix} B_1 \otimes I_{\mathcal{K}} \\ \vdots \\ B_d \otimes I_{\mathcal{K}} \end{bmatrix} \right),$$

$$\tilde{C}_n = \mathrm{blockdiag}_{i=1,\dots,d^n}\left(\begin{bmatrix} C \otimes I_{\mathcal{K}} \end{bmatrix} \right), \quad \tilde{D}_n = \mathrm{blockdiag}_{i=1,\dots,d^n}\left(\begin{bmatrix} D \otimes I_{\mathcal{K}} \end{bmatrix} \right).$$

Noncommutative nD systems

Identify

$$\mathcal{U}^{(n^d)} = \bigoplus_{\alpha \in \mathcal{F}_d, \mathrm{length}(\alpha) = n} \mathcal{U}$$

and untangle the equations.

Main example revisited (2)

Take $\mathcal{A} = \mathcal{L}(\mathcal{U})$, $E = \mathcal{L}(\mathcal{U}, \mathcal{U}^d)$, $\sigma(A) = A \otimes I_{\mathcal{K}} \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K})$. Then a dissipative system matrix has the form

$$\begin{bmatrix} A_1 \otimes I_{\mathcal{K}} & B_1 \otimes I_{\mathcal{K}} \\ \vdots & \vdots \\ A_d \otimes I_{\mathcal{K}} & B_d \otimes I_{\mathcal{K}} \\ C \otimes I_{\mathcal{K}} & D \otimes I_{\mathcal{K}} \end{bmatrix} \cong \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{U} \end{bmatrix},$$

corresponding to a system

$$\Sigma := \left\{ \begin{array}{rcl} x(n+1) & = & \tilde{A}_n x(n) + \tilde{B}_n u(n), \\ y(n) & = & \tilde{C}_n x(n) + \tilde{D}_n u(n), \end{array} \right. \qquad (n \in \mathbb{Z}_+)$$

with inputs, outputs and states $u(n), y(n) \in (\mathcal{U} \otimes \mathcal{K})^{(d^n)}$, $x(n) \in (\mathcal{X} \otimes \mathcal{K})^{(d^n)}$ and where

$$\tilde{A}_n = \mathrm{blockdiag}_{i=1,\dots,d^n} \left(\begin{bmatrix} A_1 \otimes I_{\mathcal{K}} \\ \vdots \\ A_d \otimes I_{\mathcal{K}} \end{bmatrix} \right), \quad \tilde{B}_n = \mathrm{blockdiag}_{i=1,\dots,d^n} \left(\begin{bmatrix} B_1 \otimes I_{\mathcal{K}} \\ \vdots \\ B_d \otimes I_{\mathcal{K}} \end{bmatrix} \right),$$

$$\tilde{C}_n = \mathrm{blockdiag}_{i=1,\dots,d^n} \left(\begin{bmatrix} C \otimes I_{\mathcal{K}} \end{bmatrix} \right), \quad \tilde{D}_n = \mathrm{blockdiag}_{i=1,\dots,d^n} \left(\begin{bmatrix} D \otimes I_{\mathcal{K}} \end{bmatrix} \right).$$

Commutative nD systems

Also symmetrize

$$u(\mathbf{n}) = \sum_{\alpha \in \mathcal{F}_d, \mathbf{a}(\alpha) = \mathbf{n}} u(\alpha) \quad (\mathbf{n} \in \mathbb{Z}_+^d)$$

via the abelianization map

$$\mathbf{a}(\alpha) = (n_1, \dots, n_d) \text{ if letter } i_k \text{ appears } n_k \text{ times.}$$

Main example revisited (3)

The Fock space $\mathcal{F}^2(E,\sigma)$ is equal to

$$\oplus_{n\in\mathbb{Z}_+}(\mathcal{U}\oplus\mathcal{K})^{(d^n)} = \oplus_{n\in\mathbb{Z}_+}\ell^2_{\mathcal{U}\oplus\mathcal{K}}(\text{words of length } n) = \ell^2_{\mathcal{U}\oplus\mathcal{K}}(\mathcal{F}_d).$$

Main example revisited (3)

The Fock space $\mathcal{F}^2(E, \sigma)$ is equal to

$$\oplus_{n\in\mathbb{Z}_+}(\mathcal{U} \oplus \mathcal{K})^{(d^n)} = \oplus_{n\in\mathbb{Z}_+}\ell^2_{\mathcal{U}\oplus\mathcal{K}}(\text{words of length } n) = \ell^2_{\mathcal{U}\oplus\mathcal{K}}(\mathcal{F}_d).$$

It follows that if $x(0) = 0$ and $u = (u(n))_{n\in\mathbb{Z}_+} \cong (u(\alpha))_{\alpha\in\mathcal{F}_d} \in \ell^2_{\mathcal{U}\otimes\mathcal{K}}(\mathcal{F}_d)$, then $y = (y(n))_{n\in\mathbb{Z}_+} \cong (y(\alpha))_{\alpha\in\mathcal{F}_d} \in \ell^2_{\mathcal{U}\otimes\mathcal{K}}(\mathcal{F}_d)$ and after the Z-transform

$$\hat{y}(\mathbf{T}) = T_\Sigma(\mathbf{T})\hat{u}(\mathbf{T}),$$

where $\mathbf{T} = \begin{bmatrix} T_1 & \cdots & T_d \end{bmatrix}$ is in

$$\mathbb{D}((E^\sigma)^*) \cong \{\mathbf{T} = \begin{bmatrix} T_1 & \cdots & T_d \end{bmatrix} : T_k \in \mathcal{L}(\mathcal{K}),\ \|\mathbf{T}\| < 1\}.$$

Here the transfer function T_Σ and Z-transforms $\hat{u}$ and $\hat{y}$ are given by

$$T_\Sigma(\mathbf{T}) = D \otimes I_\mathcal{K} + C \otimes I_\mathcal{K}(I - \textstyle\sum_{k=1}^d A_k \otimes T_k)^{-1}(\sum_{i=1}^d A_i \otimes T_i),$$

$$\hat{u}(\mathbf{T}) = \textstyle\sum_{\alpha\in\mathcal{F}_d}(I_\mathcal{U} \otimes \mathbf{T}^\alpha)u(\alpha), \quad \hat{y}(\mathbf{T}) = \sum_{\alpha\in\mathcal{F}_d}(I_\mathcal{U} \otimes \mathbf{T}^\alpha)y(\alpha),$$

where $\mathbf{T}^\alpha = T_{k_n} \cdots T_{k_1}$ in case $\alpha = e_{k_n} \cdots e_{k_1} \in \mathcal{F}_d$.

Main example revisited (3)

The Fock space $\mathcal{F}^2(E,\sigma)$ is equal to

$$\oplus_{n\in\mathbb{Z}_+}(\mathcal{U}\oplus\mathcal{K})^{(d^n)} = \oplus_{n\in\mathbb{Z}_+}\ell^2_{\mathcal{U}\oplus\mathcal{K}}(\text{words of length } n) = \ell^2_{\mathcal{U}\oplus\mathcal{K}}(\mathcal{F}_d).$$

It follows that if $x(0)=0$ and $u=(u(n))_{n\in\mathbb{Z}_+}\cong(u(\alpha))_{\alpha\in\mathcal{F}_d}\in\ell^2_{\mathcal{U}\otimes\mathcal{K}}(\mathcal{F}_d)$, then $y=(y(n))_{n\in\mathbb{Z}_+}\cong(y(\alpha))_{\alpha\in\mathcal{F}_d}\in\ell^2_{\mathcal{U}\otimes\mathcal{K}}(\mathcal{F}_d)$ and after the Z-transform

$$\hat{y}(\mathbf{T}) = T_\Sigma(\mathbf{T})\hat{u}(\mathbf{T}),$$

where $\mathbf{T}=\begin{bmatrix} T_1 & \cdots & T_d\end{bmatrix}$ is in

$$\mathbb{D}((E^\sigma)^*)\cong\{\mathbf{T}=\begin{bmatrix} T_1 & \cdots & T_d\end{bmatrix} : T_k\in\mathcal{L}(\mathcal{K}),\ \|\mathbf{T}\|<1\}.$$

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$$T_\Sigma(\mathbf{T}) = D\otimes I_\mathcal{K} + C\otimes I_\mathcal{K}(I-\textstyle\sum_{k=1}^d A_k\otimes T_k)^{-1}(\sum_{i=1}^d A_i\otimes T_i),$$

$$\hat{u}(\mathbf{T}) = \textstyle\sum_{\alpha\in\mathcal{F}_d}(I_\mathcal{U}\otimes\mathbf{T}^\alpha)u(\alpha), \quad \hat{y}(\mathbf{T}) = \sum_{\alpha\in\mathcal{F}_d}(I_\mathcal{U}\otimes\mathbf{T}^\alpha)y(\alpha),$$

where $\mathbf{T}^\alpha = T_{k_n}\cdots T_{k_1}$ in case $\alpha = e_{k_n}\cdots e_{k_1}\in\mathcal{F}_d$.

Noncommutative nD systems ($\mathcal{K}=\ell^2(\mathbb{Z})$):

Identify $T_1,\dots,T_d\in\mathcal{L}(\ell^2(\mathbb{Z}))$ with noncommutative indeterminates.

Main example revisited (3)

The Fock space $\mathcal{F}^2(E,\sigma)$ is equal to

$$\oplus_{n\in\mathbb{Z}_+}(\mathcal{U}\oplus\mathcal{K})^{(d^n)} = \oplus_{n\in\mathbb{Z}_+}\ell^2_{\mathcal{U}\oplus\mathcal{K}}(\text{words of length } n) = \ell^2_{\mathcal{U}\oplus\mathcal{K}}(\mathcal{F}_d).$$

It follows that if $x(0)=0$ and $u=(u(n))_{n\in\mathbb{Z}_+}\cong(u(\alpha))_{\alpha\in\mathcal{F}_d}\in\ell^2_{\mathcal{U}\otimes\mathcal{K}}(\mathcal{F}_d)$, then $y=(y(n))_{n\in\mathbb{Z}_+}\cong(y(\alpha))_{\alpha\in\mathcal{F}_d}\in\ell^2_{\mathcal{U}\otimes\mathcal{K}}(\mathcal{F}_d)$ and after the Z-transform

$$\hat{y}(\mathbf{T}) = T_\Sigma(\mathbf{T})\hat{u}(\mathbf{T}),$$

where $\mathbf{T}=\begin{bmatrix} T_1 & \cdots & T_d\end{bmatrix}$ is in

$$\mathbb{D}((E^\sigma)^*)\cong\{\mathbf{T}=\begin{bmatrix} T_1 & \cdots & T_d\end{bmatrix} : T_k\in\mathcal{L}(\mathcal{K}),\ \|\mathbf{T}\|<1\}.$$

Here the transfer function T_Σ and Z-transforms $\hat{u}$ and $\hat{y}$ are given by

$$T_\Sigma(\mathbf{T}) = D\otimes I_\mathcal{K} + C\otimes I_\mathcal{K}(I-\textstyle\sum_{k=1}^d A_k\otimes T_k)^{-1}(\sum_{i=1}^d A_i\otimes T_i),$$

$$\hat{u}(\mathbf{T}) = \textstyle\sum_{\alpha\in\mathcal{F}_d}(I_\mathcal{U}\otimes\mathbf{T}^\alpha)u(\alpha),\quad \hat{y}(\mathbf{T}) = \sum_{\alpha\in\mathcal{F}_d}(I_\mathcal{U}\otimes\mathbf{T}^\alpha)y(\alpha),$$

where $\mathbf{T}^\alpha = T_{k_n}\cdots T_{k_1}$ in case $\alpha = e_{k_n}\cdots e_{k_1}\in\mathcal{F}_d$.

Commutative nD systems ($\mathcal{K}=\mathbb{C}$):

Because the entries in $\mathbf{T}=(z_1,\dots,z_d)\in\mathbb{D}((E^\sigma)^*)\subset\mathbb{C}^d$ commute

$$\hat{u}(z_1,\dots,z_d) = \sum_{\mathbf{n}\in\mathbb{Z}^d_+} z^{\mathbf{n}}u(\mathbf{n})$$

with $u(\mathbf{n})$ defined via symmetrization.

THANKS FOR YOUR ATTENTION